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Entanglement sudden death in a spin channel

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Abstract
We study the dynamic evolution of the entanglement in a spin channel with an XY-type interaction initially in an entangled state. We find that the phenomenon of entanglement sudden death (ESD) appears in the evolution of entanglement for some initial states. We calculate the entanglement and obtain the parameter regions of disentanglement for the chains with several numbers of sites. The influence of the presence of one common spin environment is also discussed.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

As promising candidates, spin systems have been proposed to be used in many quantum information processes [1]. Spin chains have also been considered as quantum 'wires' for the quantum information transfer. S Bose [2] proposed a scheme to use a spin chain as a channel for short distance quantum communication. The communication is achieved by placing a spin state at one end of the chain A (Alice) and waiting for a specific amount of time to let this state propagate to the other end B (Bob) [2]. Afterward, several projects of spin channels were proposed by other authors [3–9]. Christandl *et al* [3] demonstrated that a perfect transfer can be achieved via a special spin chain with mirror symmetric coupling strength, and Cai *et al* [9] discussed the decoherence effect on the quantum spin channels. The time evolution of the entangled state of two sites i and $i + r$ in a long Heisenberg chain has also been studied by Pratt and Eberly [10]. The decoherence effects were found to be undulated in their temporal behavior, and be with respect to the initial symmetry and the parity of the distance r [10].

In this paper, we draw attention to the dynamic properties of entanglement in these spin channels. Consider a spin chain with XY-type interaction, whose Hamiltonian is

$$H = \frac{J}{2}\sum_{i}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y\right). \tag{1}$$

Here N is the number of sites, J depends on the coupling strength between two nearest neighbor sites, and σ_i denotes the Pauli operator of site i. We assume that the system is prepared first in a partially entangled initial state

$$|\Psi_0\rangle = |\psi\rangle_{AB} \otimes |0\rangle^{N-2}, \tag{2}$$

where

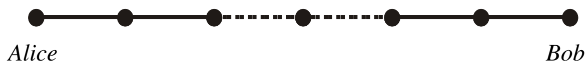
$$|\psi\rangle_{AB} = \cos\alpha|11\rangle + \sin\alpha|00\rangle, \tag{3}$$

and

$$|0\rangle^{N-2} = |0\rangle_2 \otimes |0\rangle_3 \ldots |0\rangle_{N-1}. \tag{4}$$

$|0\rangle$ ($|1\rangle$) denotes the spin down (up) state (i.e., the spin aligned along the z-direction) of a spin. It means that AB is initially entangled. Then, we will study the time evolution of the entanglement between A and B when subjected to Hamiltonian (1) for the chains with several numbers of sites.

Yu and Eberly [11] have studied the dynamics of bipartite entanglement between two atoms which couple to their own dissipative environment respectively. They found that the entanglement can completely vanish in a finite time, termed entanglement sudden death (ESD). This surprising phenomenon which is contrary to our intuition on the decoherence also appears in many other scenarios [12–14]. The disentanglement of two initially entangled Jaynes–Cummings atoms without any interaction between them has been studied in [12]; the effects of interaction between the particles and the couplings to the same environment have been discussed in [13, 14]; and the ESD is demonstrated can also happen in closed systems [15]. Although extensively works have been done in ESD, it is still unclear what causes it and what is the physics behind.



2. Measure of entanglement

To calculate the entanglement between Alice and Bob, we choose the concurrence C defined by Wootters [16] as a convenient measure of entanglement. The concurrence varies from $C = 0$ of a separable state to $C = 1$ of a maximally entangled state. For a pure or mixed state of two qubits A and B, the concurrence may be calculated explicitly from the density matrix ρ as

$$C(\rho) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}), \tag{5}$$

where the quantities λ_i are the eigenvalues in decreasing order of the matrix

$$\zeta = \rho\left(\sigma_y^A \otimes \sigma_y^B\right)\rho^*\left(\sigma_y^A \otimes \sigma_y^B\right), \tag{6}$$

where ρ^* denotes the complex conjugation of ρ in the standard basis and σ_y is the Pauli matrix expressed in the same basis as

$$\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{7}$$

3. Entanglement sudden death

The Hamiltonian in the equation (1) can be written as

$$H_N = J \sum_{i}^{N-1} \left(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+\right), \tag{8}$$

where σ^+ and σ^- are, respectively, the spin raising and lowering operators.

We first consider the chain with the number of sites $N = 4$, the initial state $|\Psi_0\rangle$ in the equation (2) can be written as

$$|\Psi_0\rangle = \sin\alpha|\psi_1\rangle + \cos\alpha\left[\tfrac{1}{\sqrt{5}}|\psi_7\rangle + \tfrac{1}{\sqrt{5}}|\psi_8\rangle - \tfrac{\sqrt{3}}{5}|\psi_2\rangle - \tfrac{2\sqrt{2}}{5}|\psi_3\rangle\right], \tag{9}$$

where

$$\begin{aligned}
|\psi_1\rangle &= |0000\rangle,\\
|\psi_2\rangle &= \tfrac{1}{\sqrt{3}}(|1100\rangle - |1001\rangle + |0011\rangle),\\
|\psi_3\rangle &= \tfrac{1}{\sqrt{2}}(|0110\rangle - |1001\rangle),\\
|\psi_7\rangle &= \tfrac{1}{\sqrt{20}}|1100\rangle - \tfrac{1}{2}|1010\rangle + \tfrac{1}{\sqrt{5}}|1001\rangle + \tfrac{1}{\sqrt{5}}|0110\rangle - \tfrac{1}{2}|0101\rangle + \tfrac{1}{\sqrt{20}}|0011\rangle,\\
|\psi_8\rangle &= \tfrac{1}{\sqrt{20}}|1100\rangle + \tfrac{1}{2}|1010\rangle + \tfrac{1}{\sqrt{5}}|1001\rangle + \tfrac{1}{\sqrt{5}}|0110\rangle + \tfrac{1}{2}|0101\rangle + \tfrac{1}{\sqrt{20}}|0011\rangle
\end{aligned} \tag{10}$$

are the eigenstates of H_4, and the corresponding eigenenergies are $E_1 = E_2 = E_3 = 0$, $E_7 = -\sqrt{5}J$, $E_8 = \sqrt{5}J$. Thus we can obtain the time evolution of $|\Psi_0\rangle$ as

$$|\Psi(t)\rangle = \sin\alpha|\psi_1\rangle + \cos\alpha\left[\tfrac{1}{\sqrt{5}}\,\mathrm{e}^{\sqrt{5}iJt}|\psi_7\rangle + \tfrac{1}{\sqrt{5}}\,\mathrm{e}^{-\sqrt{5}iJt}|\psi_8\rangle - \tfrac{\sqrt{3}}{5}|\psi_2\rangle - \tfrac{2\sqrt{2}}{5}|\psi_3\rangle\right]. \tag{11}$$

To calculate the entanglement of AB, we trace out all the sites except A and B, and obtain the reduced density matrix of AB as

$$\rho_{AB} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14}\\ 0 & \rho_{22} & 0 & 0\\ 0 & 0 & \rho_{33} & 0\\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned}
\rho_{11} &= \tfrac{1}{25}[2\cos(\sqrt{5}Jt) + 3]^2\cos^2\alpha,\\
\rho_{22} &= \rho_{33} = \tfrac{1}{25}([\cos(\sqrt{5}Jt) - 1]^2 + 5\sin^2(\sqrt{5}Jt))\cos^2\alpha,\\
\rho_{44} &= 1 + \tfrac{4}{25}[\cos(\sqrt{5}Jt) - 1]^2,\\
\rho_{14} &= \rho 41 = \tfrac{1}{5}[3 + 2\cos(\sqrt{5}Jt)]\cos\alpha\cdot\sin\alpha.
\end{aligned} \tag{13}$$

We can then obtain the concurrence

$$C(t) = 2\,\mathrm{Max}[0, f(t)], \tag{14}$$

where

$$f(t) = \tfrac{1}{5}[3 + 2\cos(\sqrt{5}Jt)]|\cos\alpha\cdot\sin\alpha| - \tfrac{1}{25}([\cos(\sqrt{5}Jt) - 1]^2 + 5\sin^2(\sqrt{5}Jt))\cos^2\alpha. \tag{15}$$

We find that the evolution of entanglement exhibits different properties for different initial states. The entanglement evolution curves E_{AB} for $\alpha = \pi/4$ and $\pi/6$ are plotted in figure 1; we can easily find that, for $\alpha = \pi/6$, the entanglement goes through a sudden death and retains zero for a period of time and then revives, but for $\alpha = \pi/4$ the ESD does not appear.

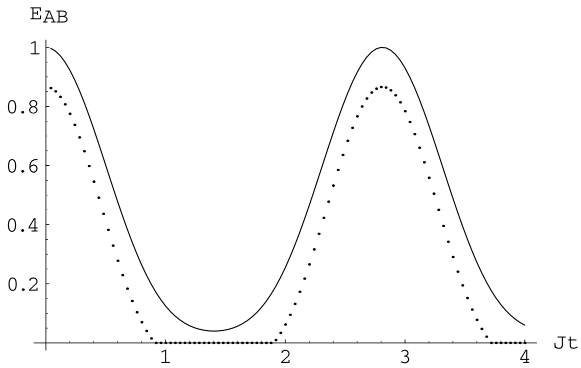


Figure 1. The entanglement E_{AB} versus scaled time Jt for $\alpha = \pi/4$ (real line) and $\alpha = \pi/6$ (dashed line).

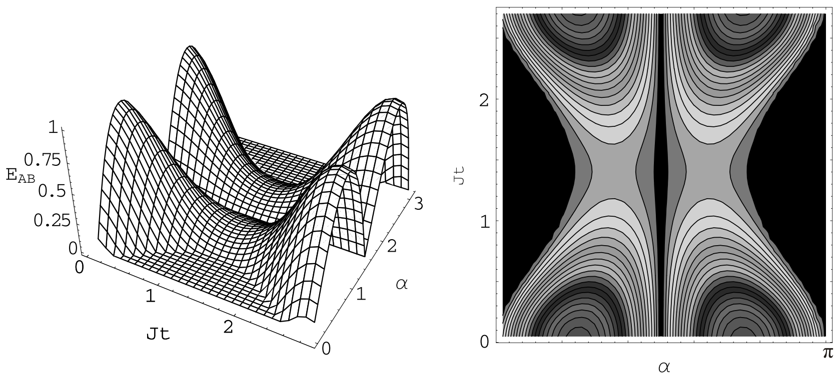


Figure 2. Schematic curve and contour plot of the Entanglement between AB as a function of scaled time and α in the chain with the number of sites $N = 4$.

Obviously, the phenomenon of ESD is correlative to the initial state, we plot the entanglement as a function of time and α in the left sub-figure of figure 2, and the contour plot is in the right sub-figure. The entanglement is symmetrical with respect to the plane of $\alpha = \frac{\pi}{2}$, when $\alpha \leqslant \alpha_0$ and $\pi - \alpha_0 \leqslant \alpha < \pi$, a sudden death appears in the evolution curve, and when $\alpha_0 < \alpha < \pi - \alpha_0$, there is no ESD. The death regions are shown by the two dark regions. From equations (14) and (15), we can easily obtain the parameter regions of disentanglement, the critical point $\alpha_0 = \mathrm{Arctan}\left(\frac{4}{5}\right)$.

We also calculate numerically the entanglement of chains with a larger number of sites. In figures 3 and 4, we present the schematic curves of the evolution of E_{AB} for $N = 5$ and $N = 6$ respectively. The evolution of E_{AB} with $N = 5$ is more complicated than that with $N = 4$, as seen in figure 3, two entanglement islands appear in the two disentanglement regions and divide them into four disentanglement regions, and the ESD appears in the entanglement evolution for any α. For the chain with $N = 6$, the entanglement curves are similar to the $N = 4$, and we obtain $\alpha_0 = 0.78$.

To compare the evolution of entanglement for different numbers of sites, we plot the schematic curves for $\alpha = \frac{\pi}{6}$, seen in figure 5. We can find that the time of reviving increases with the increase in the number of sites N, and the behaviors of curves are different for even

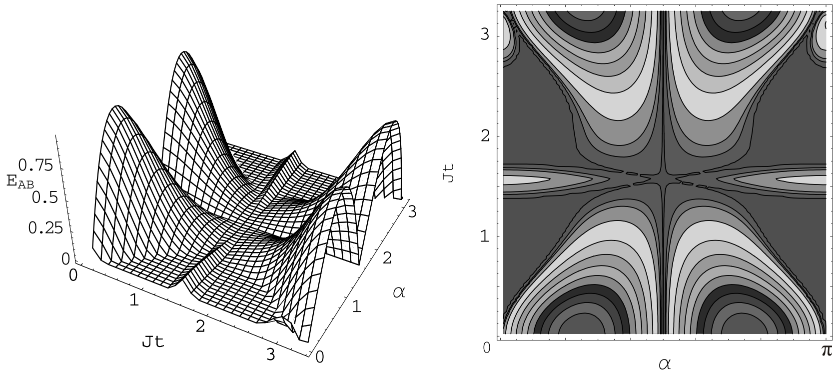


Figure 3. Schematic curve and contour plot of the evolution of E_{AB} in the chain with the number of sites $N = 5$.

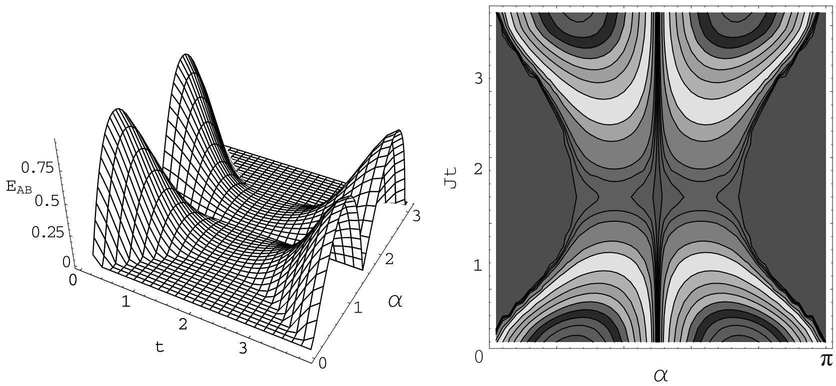


Figure 4. Schematic curve and contour plot of the evolution of E_{AB} in the chain with the number of sites $N = 6$.

and odd N. For the even number of sites, the entanglement falls to zero and retains zero for a period of time and then revives, but for the odd number of sites, there is an entanglement protuberance appearing in the disentanglement region.

4. Another type of initial state

Consider another type of initial entangled state

$$|\Phi_0\rangle = |\phi\rangle_{AB} \otimes |0\rangle^{N-2}, \tag{16}$$

where

$$|\phi\rangle_{AB} = \cos\alpha|10\rangle + \sin\alpha|01\rangle. \tag{17}$$

Our numerical calculation reveals that there is no sudden death of E_{AB} in the evolution of this type of initial state. The result of the chain with $N = 4$ is plotted in figure 6.

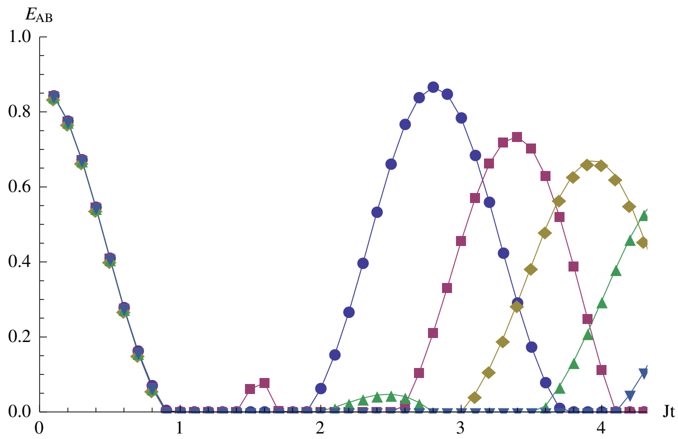


Figure 5. Schematic curves for $\alpha = \frac{\pi}{6}$ with the number of sites $N = 4$ (dot), $N = 5$ (square), $N = 6$ (diamond), $N = 7$ (up triangle), $N = 8$ (down triangle).

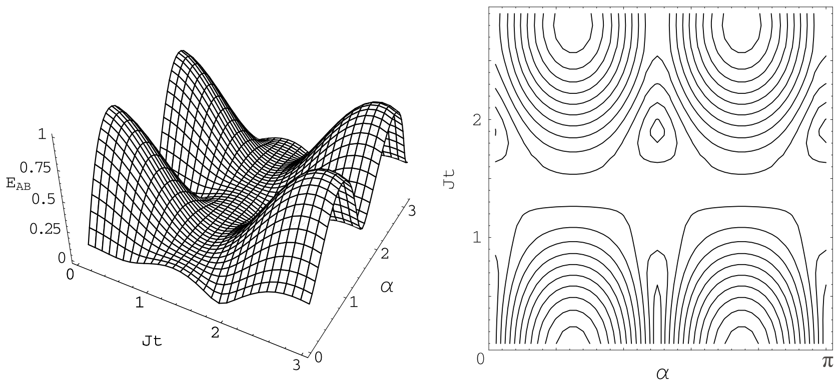


Figure 6. Schematic curve and contour plot of the evolution of E_{AB} in the chain with the number of sites $N = 4$.

5. Effect of environment

The real physical systems, especially a solid-state system, are inevitably in a noise environment and influenced by it [17]. We then discuss the entanglement dynamics with the presence of one common spin environment [9, 18], which is formed by M independent spins. The whole system of the spin chain S and the environment ε is described by the Hamiltonian

$$H_T = H_N + H_{S\varepsilon}, \tag{18}$$

where

$$H_{S\varepsilon} = \frac{1}{2}\sum_{i=1}^{N}\sigma_i^z \otimes \sum_{k=1}^{M} g_k\sigma_k^z \tag{19}$$

is the coupling between the spin chain and the environment, and the self-Hamiltonian of the environment ε is neglected.

We denote the basis of the environment $\{|m\rangle\langle m|\}$, $\sum_{m=0}^{2^M-1}|m\rangle\langle m| = I_\varepsilon$, where $|m\rangle = |m_1 m_2, \ldots, m_M\rangle$ with $\sigma_k^z|m_k\rangle = (-1)^{m_k}|m_k\rangle$. Following the discussion in [9], when the environment is in an arbitrary pure state $\sum_{m=0}^{2^M-1} c_m|m\rangle$, we can obtain the pariwise-reduced density matrix of AB for the number of sites $N = 4$ as

$$\rho_{AB} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14}\gamma(t) \\ 0 & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 \\ \rho_{44}\gamma^*(t) & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{20}$$

where

$$\gamma(t) = \sum_{m=1}^{2^M-1} |c_m|^2 \,\mathrm{e}^{2\mathrm{i}B_m t}, \tag{21}$$

and

$$B_m = \sum_{k=1}^{M} (-1)^{m_k} g_k. \tag{22}$$

To obtain the result we rewrite [9, 18] equation (21) as

$$\gamma(t) = \int \mathrm{e}^{-2\mathrm{i}Bt}\eta(B)\,\mathrm{d}B. \tag{23}$$

The factor is the Fourier transform of a characteristic function

$$\eta(B) = \sum_{m=0}^{2^M-1} |c_m|^2 \delta(B - B_m). \tag{24}$$

For a general spin environment with large M, the character function $\eta(B)$ is approximately Gaussian [9, 18], that is $\eta(B) = \exp(-B^2/\sigma)/\sqrt{\pi\sigma}$. We can obtain

$$\gamma(t) = \mathrm{e}^{-\sigma t^2}. \tag{25}$$

The entanglement between AB is

$$C'(t) = 2\,\mathrm{Max}[0, f'(t)] \tag{26}$$

that

$$f'(t) = \tfrac{1}{5}[3 + 2\cos(\sqrt{5}Jt)]|\cos\alpha \cdot \sin\alpha|\,\mathrm{e}^{-2\sigma t^2} \\ - \tfrac{1}{25}([\cos(\sqrt{5}Jt) - 1]^2 + 5\sin^2(\sqrt{5}Jt))\cos^2\alpha. \tag{27}$$

To compare with the result of the closed system in section 3, we plot the evolution curves for the initial $\alpha = \frac{\pi}{4}$ in the chain of $N = 4$ with $\sigma = 0.02J^2$ (green line), $\sigma = 0.1J^2$ (blue line), $\sigma = 0.5J^2$ (red line) in figure 7, and the no disentanglement black line denotes the closed system with no environment. We can find that, with the presence of the environment, the disentanglement appears, and with the increase in σ, the decoherence becomes faster and the entanglement vanishes after several weak revivals. A schematic curve and a contour plot with $\sigma = 0.1J^2$ are in figure 8, the entanglement vanishes after a weak revival, and the regions of disentanglement seen as the dark regions in the right sub-figure of the figure 8 are more enlarged than those in figure 2.

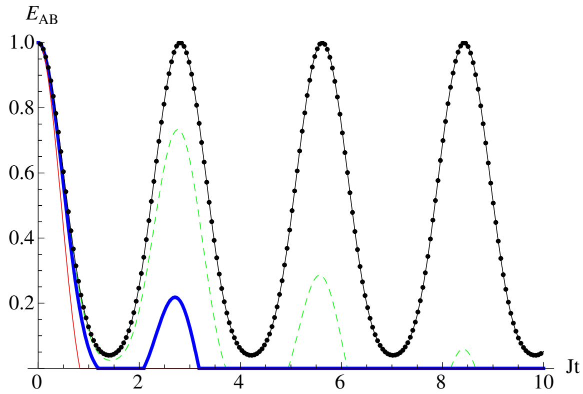


Figure 7. Schematic curves of the evolution of E_{AB} for the initial $\alpha = \frac{\pi}{4}$ in the chain of $N = 4$ with the $\sigma = 0.02J^2$ (dashed line), $\sigma = 0.1J^2$ (thick solid line), $\sigma = 0.5J^2$ (thin solid line) and with no environment (dotted line).

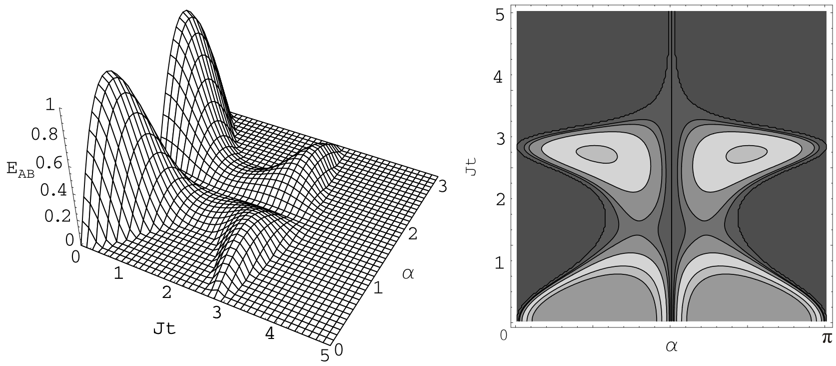


Figure 8. Schematic curve and contour plot of the evolution of E_{AB} in the chain of $N = 4$ with $\sigma = 0.1J^2$.

6. Conclusion

In this paper, we discuss the dynamic evolution of the spin channel with an XY-type interaction. For some initial states, the evolution of the entanglement between A and B undergoes an entanglement sudden death. The behaviors of entanglement curves are different for the chain with even and odd numbers of sites. We schematically give the parameter regions of the disentanglement for the situation with different numbers of sites. The effect of environment is also considered, we find that the decoherence becomes much faster and the entanglement vanishes after several weak revivals, the parameter region of disentanglement is enlarged. Whether and how the entanglement sudden death influences the quantum information processing in spin and other channels is still unclear and needs further studies.

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